

Topological acceleration in relativistic cosmology

Boudewijn F. Roukema

Toruń Centre for Astronomy, Nicolaus Copernicus University, ul. Gagarina 11
87-100 Toruń, Poland

PACS: 98.80.-k, 98.80.Es, 98.80.Jk

Keywords: cosmology: observations – cosmology: theory – cosmic microwave background

Abstract

Heuristic approaches in cosmology bypass more difficult calculations that would more strictly agree with the standard Einstein equation. These give us the well-known Friedmann-Lemaître-Robertson-Walker (FLRW) models, and, more recently, the feedback effect of the global topology of spatial sections on the acceleration of test particles. Forcing the FLRW heuristic model on observations leads to dark energy, which, pending fully relativistic calculations, is best interpreted as an artefact. Could topological acceleration also be an artefact of using a heuristic approach? A multiply connected exact solution of the Einstein equation shows that topological acceleration is present in at least one fully relativistic case—it is not an artefact of Newtonian-like thinking.

1 Can we go beyond heuristics?

A spatial section of the present-day Universe is clearly inhomogeneous, i.e. the density $\rho(\mathbf{x}, t)$ over the spatial section (3-manifold) at a given cosmological time t cannot be written as a function of t alone. But finding a general, inhomogeneous solution to the Einstein field equation is difficult. This is why the standard family of cosmological models consists of the Friedmann-Lemaître-Robertson-Walker (FLRW) models with constant curvature at any given cosmological time t , i.e. $\rho(\mathbf{x}, t) = \rho(t)$, and a fixed 3-manifold topology consistent with the comoving curvature (de Sitter, 1917; Friedmann, 1923, 1924; Lemaître, 1927; Robertson, 1935). Strictly speaking, the models are inconsistent with the existence of galaxies. This is bypassed by the heuristic approach of (mathematically) hypothesising that perturbations on the exact solution can be evolved in such a way that the perturbed global spacetime remains a so-

lution of the Einstein equation. The Concordance Model estimates of the FLRW model parameters (Ostriker and Steinhardt, 1995, , refs therein, and citations thereof) show that forcing the FLRW model on extragalactic observations requires the existence of a cosmological constant or dark energy parameter Ω_Λ , which becomes significant at approximately the epoch when overdensities become significantly non-linear. Work towards more realistic, physical models indicates that the simplest interpretation of Ω_Λ is that it is an artefact of forcing an oversimplified model (the FLRW model) onto real world data (e.g. Célérier et al., 2010; Buchert and Carfora, 2003; Wiegand and Buchert, 2010).

A complementary heuristic calculation is the effect of the topology of the spatial section ((Lachièze-Rey and Luminet, 1995; Luminet, 1998; Starkman, 1998; Luminet and Roukema, 1999; Rebouças and Gomero, 2004); or shorter: (Roukema, 2000)) on the dynamics. Again, in a strictly FLRW model, there is no effect of topology on dynamics (apart from the link between the sign of curvature and the family of constant curvature 3-manifolds). Since the real Universe is inhomogeneous, the first way of determining the nature of the effect of topology on dynamics was heuristic. The acceleration of a test particle towards a single point-like massive object in a homogeneous background was calculated in Roukema et al. (2007) for a slab space (one multiply-connected direction with zero curvature, T^1) and a 3-torus (T^3) model. Topological acceleration was found to exist at linear order in the (small) displacement of the test particle from the massive particle for the T^1 case and the irregular T^3 case (unequal fundamental lengths). In a slab space of injectivity diameter (“box size”) L , if we place the test particle along the line joining the multiple images of the massive object of mass M ,

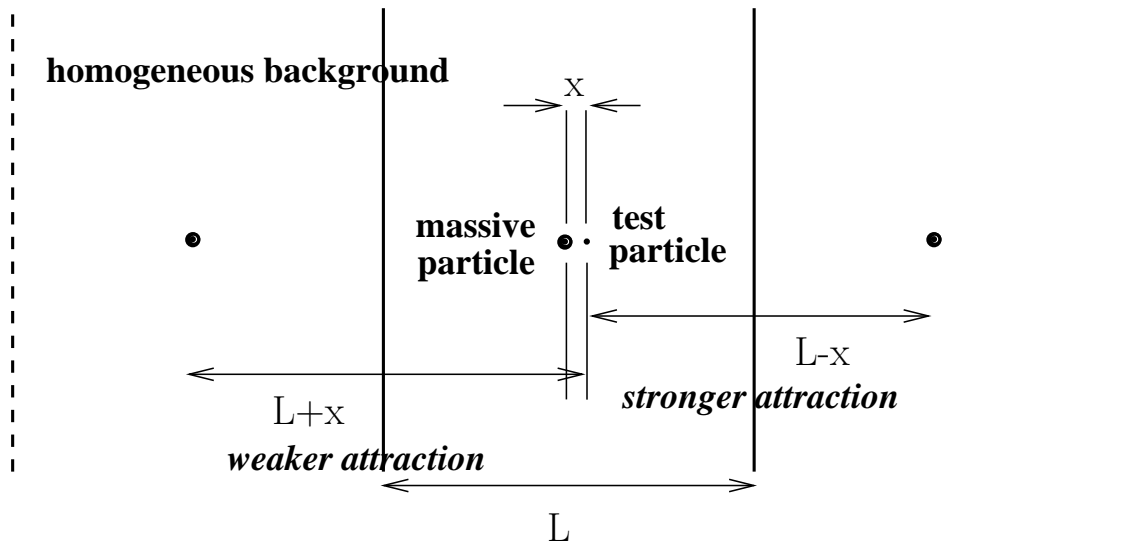


Figure 1: Two-dimensional cut through a T^1 (slab space) model, showing the “zeroth” copy of the massive object and adjacent copies in the universal covering space at left and right, separated from the first copy by the fundamental domain length L . If the third spatial dimension is ignored, this can be thought of as an ordinary cylinder. A massless test particle at distance x from the zeroth copy of the massive particle is shown.

at a small separation x from a copy of the massive object, then the test particle is at $L - x$ from one “adjacent distant” copy of the massive particle (in the universal covering space) and at $L + x$ from the other “adjacent distant” copy of the massive particle (see Fig. 1). Thus, the pseudo-Newtonian topological acceleration (ignoring the attraction towards the close copy of the massive object) is (Roukema et al., 2007)

$$\begin{aligned} \ddot{x}_{\text{topo}} &= M \left[\frac{1}{(L-x)^2} - \frac{1}{(L+x)^2} \right] \\ &\approx \frac{4M}{L^2} \frac{x}{L} \end{aligned} \quad (1)$$

to first order in $x/L \ll 1$, where the gravitational constant is in natural units ($G = 1$). Inclusion of more distant copies of the massive object without bound gives a slightly higher value:

$$\begin{aligned} \ddot{x}_{\text{topo}} &= M \sum_{j=1}^{\infty} \left[\frac{1}{(jL-x)^2} - \frac{1}{(jL+x)^2} \right] \\ &= 4\zeta(3) \frac{M}{L^2} \frac{x}{L} \end{aligned} \quad (2)$$

using Apéry’s constant $\zeta(3)$, where ζ is the Riemann zeta function.

Curiously, topological acceleration, calculated according to this heuristical approach, cancels to first order in a regular T^3 model (Roukema et al., 2007). In the three spherical 3-manifolds with the most symmetrical properties, i.e. the “well-proportioned” ones (Weeks et al., 2004), topological acceleration also cancels to first order (Roukema and Róžański, 2009). In two of them, the octahedral space S^3/T^* and the truncated-cube space S^3/O^* (Gausmann et al., 2001), the effect reappears at third order, as it does for the regular T^3 model. In the third space, the Poincaré dodecahedral space S^3/I^* , the cubical term also cancels exactly, leaving a fifth order term. Thus, topological acceleration not only shows that global topology has an effect on the dynamics of a universe, but that it distinguishes different topologies. The particular way that it distinguishes different 3-manifolds happens to select the Poincaré space as being special—not just “well balanced”, but “very well balanced”—independently of the cosmic microwave background observational arguments first presented several years earlier in favour of the Poincaré space (Luminet et al., 2003; Aurich et al., 2005a,b; Gundermann, 2005; Caillerie et al., 2007; Roukema et al., 2008a,b).

Yet, just as we lack evidence for “dark energy” (and acceleration of the scale factor $a(t)$) being any-

thing more than an artefact of a heuristical approach to solving the Einstein equation, could the topological acceleration calculations (Roukema et al., 2007; Roukema and Róžański, 2009) also be an artefact of a heuristical, i.e. pseudo-Newtonian approach?

2 Compact Schwarzschild spacetime

An exact Schwarzschild-like solution of the Einstein equations is known for which the spatial topology (apart from the “central” black hole singularity itself) is the slab space (Korotkin and Nicolai, 1994). For a test particle distant from the event horizon and an injectivity diameter (L) which is much greater still, it should be possible to check whether Eq. (2) is a limiting case of this exact, “compact Schwarzschild” solution.

This was checked in Ostrowski et al. (2012). Using the conventions $G = c = 1$, a signature of $(-, +, +, +)$, Greek spacetime indices $0, 1, 2, 3$ and Roman space indices $1, 2, 3$, the following assumptions can be made (Ostrowski et al., 2012). The test particle is distant from the event horizon but much closer to the “zeroth” copy of the massive object than to any other copies in the universal covering space:

$$0 < M \ll x \ll L/2; \quad (3)$$

the test particle has a low coordinate velocity (implying a low 4-velocity spatial component):

$$\frac{dx^i}{dt} \ll 1 \Rightarrow \frac{dx^i}{d\tau} \ll 1 \lesssim \frac{dt}{d\tau}; \quad (4)$$

and the spacetime is a vacuum model:

$$\rho = 0, \quad (5)$$

where the proper time τ along the test particle’s world line parametrises the latter, i.e. $x^\alpha(\tau)$, and $t \equiv x^0$.

Following Korotkin and Nicolai (1994), the metric is given in Weyl coordinates, using the Ernst potential:

$$ds^2 = -e^\omega dt^2 + e^{-\omega} [e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2] \quad (6)$$

from Eq. (9) of Korotkin and Nicolai (1994), where k and ω are related to an Ernst potential ε defined on $\xi := x + i\rho$, and

$$\varepsilon_0(x, \rho) := \frac{\sqrt{(x-M)^2 + \rho^2} + \sqrt{(x+M)^2 + \rho^2} - 2M}{\sqrt{(x-M)^2 + \rho^2} + \sqrt{(x+M)^2 + \rho^2} + 2M} \quad (7)$$

$$\omega_0 := \ln \varepsilon_0, \quad a_0 := 0, \quad a_{j \neq 0} := \frac{2M}{L|j|} \quad (8)$$

$$\omega(x, \rho) := \sum_{j=-\infty}^{\infty} [\omega_0(x + jL, \rho) + a_j], \quad \varepsilon := e^\omega \quad (9)$$

$$\partial_\xi k = 2i\rho \frac{\partial_\xi \varepsilon \partial_{\bar{\xi}} \bar{\varepsilon}}{(\varepsilon + \bar{\varepsilon})^2}, \quad (10)$$

i.e. Eqs (13), (12), (5) and (7) of Korotkin and Nicolai (1994). As shown in Ostrowski et al. (2012), the topological acceleration of a test particle under these conditions is identical to that of Eq. (2)

Numerical evaluation of \ddot{x}_{topo} using the Weyl metric and 100-bit arithmetic shows that for $M \sim 10^{14} M_\odot$, $L \sim 10$ to $20 h^{-1}$ Gpc, the linear expression is quite accurate over several orders of magnitude in length scale, i.e. the linear expression and the numerical estimate agree to within $\pm 10\%$ for $3h^{-1}$ Mpc $\lesssim x \lesssim 2h^{-1}$ Gpc (Ostrowski et al., 2012).

3 Conclusion

Thus, it is clear that topological acceleration is not a Newtonian artefact: it exists in at least one relativistic spacetime, and most likely in others. Moreover, given some typical astronomical values for the parameters of the model, the linear (Newtonian-like, first-order) approximation of the effect is a good estimate of the effect over several orders of magnitude in length scale. It remains to be seen if exact solutions can be found for T^3 , S^3/T^* , S^3/O^* and S^3/I^* spatial sections, and if the linear terms cancel exactly as in the pseudo-Newtonian case.

Acknowledgments

A part of this project has made use of Program Obliczeń Wielkich Wyzwań nauki i techniki (POWIEW) computational resources (grant 87) at the Poznań Supercomputing and Networking Center (PCSS).

References

- W. de Sitter, *MNRAS* **78**, 3–28 (1917).
- A. Friedmann, *Mir kak prostranstvo i vremya* (*The Universe as Space and Time*), Petrograd: Academia, 1923.

- A. Friedmann, *Zeitschr. für Phys.* **21**, 326 (1924).
- G. Lemaître, *Annales de la Société Scientifique de Bruxelles* **47**, 49–59 (1927).
- H. P. Robertson, *ApJ* **82**, 284 (1935).
- J. P. Ostriker, and P. J. Steinhardt, *ArXiv Astrophysics e-prints* (1995), [astro-ph/9505066](#).
- M. Célérier, K. Bolejko, and A. Krasinski, *A&A* **518**, A21 (2010), [0906.0905](#).
- T. Buchert, and M. Carfora, *Phys. Rev. Lett.* **90**, 031101 (2003), [gr-qc/0210045](#).
- A. Wiegand, and T. Buchert, *Phys. Rev. D.* **82**, 023523–+ (2010), [1002.3912](#).
- M. Lachièze-Rey, and J. Luminet, *Physics Reports* **254**, 135 (1995), [gr-qc/9605010](#).
- J.-P. Luminet, *Acta Cosmologica* **XXIV-1**, 105 (1998), [gr-qc/9804006](#).
- G. D. Starkman, *ClassQuantGrav* **15**, 2529–2538 (1998).
- J. Luminet, and B. F. Roukema, “Topology of the Universe: Theory and Observation,” in *NATO ASIC Proc. 541: Theoretical and Observational Cosmology. Publisher: Dordrecht: Kluwer*, edited by M. Lachièze-Rey, 1999, p. 117, [astro-ph/9901364](#).
- M. J. Rebouças, and G. I. Gomero, *Braz. J. Phys.* **34**, 1358 (2004), [astro-ph/0402324](#).
- B. F. Roukema, *BullAstrSocIndia* **28**, 483 (2000), [astro-ph/0010185](#).
- B. F. Roukema, S. Bajtlik, M. Biesiada, A. Szaniewska, and H. Jurkiewicz, *A&A* **463**, 861–871 (2007), [astro-ph/0602159](#).
- J. Weeks, J.-P. Luminet, A. Riazuelo, and R. Lehoucq, *MNRAS* **352**, 258 (2004), [astro-ph/0312312](#).
- B. F. Roukema, and P. T. Różański, *A&A* **502**, 27 (2009), [0902.3402](#).
- E. Gausmann, R. Lehoucq, J.-P. Luminet, J.-P. Uzan, and J. Weeks, *ClassQuantGrav* **18**, 5155 (2001), [gr-qc/0106033](#).
- J. Luminet, J. R. Weeks, A. Riazuelo, R. Lehoucq, and J. Uzan, *Nature* **425**, 593 (2003), [astro-ph/0310253](#).
- R. Aurich, S. Lustig, and F. Steiner, *ClassQuantGrav* **22**, 3443 (2005a), [astro-ph/0504656](#).
- R. Aurich, S. Lustig, and F. Steiner, *ClassQuantGrav* **22**, 2061 (2005b), [astro-ph/0412569](#).
- J. Gundermann, *ArXiv e-prints* (2005), [astro-ph/0503014](#).
- S. Caillerie, M. Lachièze-Rey, J.-P. Luminet, R. Lehoucq, A. Riazuelo, and J. Weeks, *A&A* **476**, 691 (2007), [0705.0217v2](#).
- B. F. Roukema, Z. Buliński, A. Szaniewska, and N. E. Gaudin, *A&A* **486**, 55 (2008a), [0801.0006](#).
- B. F. Roukema, Z. Buliński, and N. E. Gaudin, *A&A* **492**, 657 (2008b), [0807.4260](#).
- D. Korotkin, and H. Nicolai, *ArXiv General Relativity and Quantum Cosmology e-prints* (1994), [gr-qc/9403029](#).
- J. J. Ostrowski, B. F. Roukema, and Z. P. Buliński, *ClassQuantGrav* **29**, 165006 (2012), [1109.1596](#).